Lecture 4 : Motion in two dimensions I:
Co-ordinate systems, vectors and projectiles

Co-ordinate systems (fixing landmarks)

Anyone who has adventured into the wilderness or has driven in an unfamiliar neighborhood without a map is well aware of the need to carefully keep track of where they are. A simple way to do this is to identify a set of landmarks which can be used to retrace their steps. Global positioning technology (GPS - Garmin or Magellan) has made this obsolete as there are now a set of satellites which have positions that are precisely known and can be used as landmarks to determine any location on earth to an accuracy of a couple of meters or better. Any positioning system uses co-ordinate or reference systems in which distances are measured. In everyday life we almost always treat a landmark on the earth's surface as our reference location. A position on the earth surface is determined by a distance east or west and a distance north or south from this origin, for example we may consider Sparty as the origin of our co-ordinate system and measure distances east or west and north or south from Sparty. In the case of an airplane we also have to specify an altitude. Our reference altitude is usually "sea level" so that an altitude of $35,000ft$ means $35,000ft$ above sea level.

We shall first work with two dimensional co-ordinate systems, as this already allows us to treat a wide range of problems, such as projectile motion (eg. throwing a ball) and instead of east, west, north and south we shall use a distance along the x-direction and a distance along the y-axis. In fact instead of a distance which is always positive we use an "x-co-ordinate", x, which may be positive or negative and a "y-co-ordinate", y, which may also be positive or negative. Any position, $\vec{r}_P$, in our two dimensional co-ordinate system is then specified by two numbers (x, y). This position vector is defined with respect to $(x, y) = (0, 0)$, which is the origin of our co-ordinate system. This co-ordinate system is called the Cartesian co-ordinate system and was invented by René Descartes in 1637.

Polar co-ordinates

Although it is natural to define position using the x and y co-ordinates, that does not tell us the distance of P from the origin. To find the distance we need to use the Pythagoras law,

$$r^2 = x^2 + y^2 \tag{1}$$

The distance r tells us the distance from the origin to point P, and is the magnitude of the vector $\vec{r}$ but it does not tell us the direction. The correct direction is set by defining an angle θ, which is defined as the angle from the $x-axis$ in a counterclockwise direction. The vector $\vec{r}$ is a vector with magnitude r and direction determined by θ. This way of describing the position of point P is called polar co-ordinates. Using trigonometry we write equations relating the cartesian co-ordinates (x, y) to the polar co-ordinates (r, θ). That is,

$$x = rcos(\theta) \quad ; \quad y = rsin(\theta) \tag{2}$$

The velocity and acceleration are also vectors and may be written in either cartesian co-ordinates or polar co-ordinates. For example the acceleration at the earth surface may be written in cartesian co-ordinates as,

$$\vec{a} = (a_x, a_y) = (0, -9.81m/s^2) \tag{3}$$

and in polar co-ordinates as,

$$\vec{a} = (a, \theta) = (9.81m/s^2, 270^o) = (9.81m/s^2, -90^o) \tag{4}$$

Vector operations

To describe motion in two dimensions, we need to add, subtract and carry out a basic multiplication operation with vectors. For example lets consider the final position of a student who starts at Sparty and walks $10m$ east, then turns and walks $7.07m$ south west. How far (what distance) is the student from Sparty. This is a problem in vector addition, which we can write as,

$$\vec{r}_f = \vec{r}_1 + \vec{r}_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \tag{5}$$

That is, the x co-ordinate of the final position is the sum of the x co-ordinates of $\vec{r}_1 = (10, 0)m$ and $\vec{r}_2 = (-5, -5)m$ so that $\vec{r}_f = (5, -5)m$.

Sometimes we need to subtract two vectors, which is carried out as follows,

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B}) = (A_x - B_x, A_y - B_y) \tag{6}$$

For example $-\vec{r}_2 = (5, 5)m$ and $\vec{r}_1 - \vec{r}_2 = (15, 5)m$

Finally sometimes we need to multiply a vector by a constant, which is carried out as follows,

$$c\vec{A} = (cA_x, cA_y) \tag{7}$$

For example $4\vec{r}_2 = (-20, -20)m$.

Definitions of velocity and acceleration in 2-D

The definitions of velocity and acceleration are very similar to those we used for one dimensional motion:
(i) Displacement is the change in position

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i \tag{8}$$

(ii) Average Velocity is the rate of change of displacement

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} \tag{9}$$

Instantaneous velocity $\vec{v} = lim_{\Delta t\to 0}\vec{v}_{av}$. Instantaneous speed is the magnitude of the instantaneous velocity.
(iii) Average acceleration is the rate of change of instantaneous velocity.

$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} \tag{10}$$

Instantaneous acceleration $\vec{a} = lim_{\Delta t\to 0}\vec{a}_{av}$.

Projectile motion

Projectile motion is motion under the action of gravity. At very high speeds we have to consider air drag, but at low speeds we can just consider the action of gravity. The first thing to note is that if we ignore air drag the only force acting on the projectile is the force due to gravity. The only acceleration is then the acceleration due to gravity. We thus have $\vec{a} = (0, -g)$. This is motion at constant acceleration, but it is a vector motion. Treating this vector motion in 2-d is like treating two one dimensional motions. We treat the x co-ordinate motion and the y co-ordinate motion as separate one dimensional motions, that is, motion in the x-direction is described by,

$$v_x = v_{0x} + a_x t \;\; ; \Delta x = v_{0x}t + a_x t^2/2; \;\; v_x^2 = v_{0x}^2 + 2a_x\Delta x \tag{11}$$

while motion in the y-direction is described by,

$$v_y = v_{0y} + a_y t \;\; ; \Delta y = v_{0y}t + a_y t^2/2; \;\; v_y^2 = v_{0y}^2 + 2a_y\Delta y \tag{12}$$

These equations are the same as the ones we used for the constant acceleration case in one dimension.